Journal of Applied Mechanics and Technical Physics, Vol. 36, No. 2, 1995

STABILITY OF MOTION IN A PLANAR LAYER OF INHOMOGENEOUS IDEAL INCOMPRESSIBLE LIQUID HAVING FREE BOUNDARIES*

V. E. Zakhvataev

UDC 532.5

1. Lagrangian Coordinates. One formulates the motion of an inhomogeneous ideal incompressible liquid having a free boundary as follows. At the initial instant, we are given the volume of the liquid $\Omega(0)$, which has the boundary $\Gamma(0)$, together with the density and velocity pattern:

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(\mathbf{x}), \quad \operatorname{div} \mathbf{u}_0 = 0, \quad \rho|_{t=0} = \rho_0(\mathbf{x}), \quad \mathbf{x} \in \Omega(0). \tag{1.1}$$

The volume occupied at time t is $\Omega(t)$, and the boundary is $\Gamma(t)$.

It is necessary to determine $\Omega(t)$, the velocity vector u, the pressure p, and the density ρ, which satisfy the following conditions: within $\Omega(t)$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{g}; \tag{1.2}$$

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0; \tag{1.3}$$

$$\operatorname{div} \mathbf{u} = 0; \tag{1.4}$$

and at the free boundary $\Gamma(t)$

$$p = p_0 - \sigma\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big); \tag{1.5}$$

$$\frac{df}{dt}\Big|_{f=0} = 0. \tag{1.6}$$

Here g is the vector for the external mass forces, p_0 the given external pressure, R_1 and R_2 are the principal radii of curvature for normal sections of $\Gamma(t)$ (it is assumed that $R_1 > 0$, $R_2 > 0$, if the corresponding section is convex within $\Omega(t)$), $\sigma \geq 0$ surface tension, and $f(x, t) = 0$ the equation for the free boundary $\Gamma(t)$.

A major difficulty in researching nonstationary solutions with free boundaries lies in the need to derive unknown quantities (velocity, density, and pressure) in the region $\Omega(t)$, which is not known in advance and is itself one of the unknowns. Transfer to Lagrangian coordinates enables one to consider the case in a certain fixed and known region, although the equations describing the motion can become more complicated. We introduce them as the coordinates of the liquid particles at the initial instant:

$$\mathbf{x}|_{t=0} = \boldsymbol{\xi}. \tag{1.7}$$

*Research performed with financial support from the Krasnoyarsk Scientific Fund (Project Code 2F0059).

Computing Center, Siberian Division, Russian Academy of Sciences, 660036 Krasnoyarsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 2, pp. 32-38, March-April, 1995. Original article submitted May 17, 1993; revision submitted May 6, 1994.

0021-8944/95/3602-0177$12.50

Then the law of motion is defined by the solution to

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t)$$

with (1.7) as initial condition and is derived as

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t), \quad \boldsymbol{\xi} \in \Omega(0).$$

By virtue of (1.3), $\rho(x(\xi, t), t) = \rho_0(\xi)$, and after certain transformations [1] we get a treatment for the unknowns $x(\xi, t)$, $p(\xi, t) = p(x(\xi, t), t)$:

$$\mathrm{div}_\xi(M^{-1}\mathbf{x}_t) = 0, \qquad M^*(\mathbf{x}_{tt} - \mathbf{g}) + (1/\rho_0)\nabla_\xi p = 0,$$

in which M is a Jacobi reflection matrix $x = x(\xi, t)$: $M = \partial(x)/\partial(\xi)$, with M^* and M^{-1} the transposed M matrix and the one reciprocal to it, and div_ξ, ∇_ξ denote the divergence and gradient operators with respect to the variable ξ. We subsequently omit the subscript ξ.

We add the (1.5) boundary condition written in Lagrangian coordinates and the initial condition (1.1) to get the formulation in terms of Lagrangian variables.

2. Small-Perturbation Evolution Equations in Lagrangian Coordinates. Let there be known a solution to the above treatment formulated in terms of Lagrangian coordinates with the (1.1) initial conditions, which we call the basic solution.

Consider another solution (the perturbed one) having the same initial region $\Omega(0)$ and initial data

$$\tilde{\rho}_0(\boldsymbol{\xi}) = \rho_0(\boldsymbol{\xi}) + Q_0(\boldsymbol{\xi}), \quad \tilde{\mathbf{x}}_t(\boldsymbol{\xi}) = \mathbf{u}_0(\boldsymbol{\xi}) + \mathbf{U}_0(\boldsymbol{\xi}), \quad \mathrm{div}\, \mathbf{U}_0 = 0. \tag{2.1}$$

We assume for the perturbed motion that

$$\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{X}(\boldsymbol{\xi}, t), \quad \tilde{p} = p + M^{*^{-1}}\nabla p \cdot \mathbf{X} + P(\boldsymbol{\xi}, t).$$

If we assume that the initial perturbations are small, one hopes that the perturbations will be small at least for a restricted time interval, and we consider the behavior of small perturbations in the linear approximation.

We assume that the perturbations and their derivatives are small. Then we follow [1] and obtain after transformations in terms of the Lagrangian coordinates that the linear approximation is

$$\begin{aligned} &M^*\mathbf{X}_{tt} - M^*_{tt}\mathbf{X} + \nabla P/\rho_0 - Q_0\nabla p/\rho_0^2 + \\ &+(M^{*^{-1}}\nabla p \cdot \mathbf{X})\nabla\rho_0/\rho_0^2 - A\mathbf{X} = 0, \quad A = \nabla(\mathbf{g}) - (\nabla(\mathbf{g}))^*; \end{aligned} \tag{2.2}$$

$$\mathrm{div}\,(M^{-1}\mathbf{X}) = 0; \tag{2.3}$$

$$P - aR + \sigma\bar{\Delta}_\Gamma(t)R = 0; \tag{2.4}$$

$$R = b\int_0^t \mathbf{n} \cdot M^{-1}\mathbf{U}\, dt, \quad \boldsymbol{\xi} \in \Gamma(0), \quad t \geq 0. \tag{2.5}$$

Here $\bar{\Delta}_\Gamma(t)$ is the result of transforming the Laplace–Beltrami operator to Lagrangian coordinates [1], with

$$b(\boldsymbol{\xi}, t) = \frac{|\nabla f_0|}{|M^{*^{-1}}\nabla f_0|}; \quad a = -\frac{\partial p}{\partial \mathbf{n}_{\Gamma(t)}} - \sigma\Big(\frac{1}{R_1^2} + \frac{1}{R_2^2}\Big);$$

and $n = n(\xi)$ the vector for the exterior normal to $\Gamma(0)$.

For clarity we note that in Euler coordinates $R(x, t) = n_{\Gamma(t)} \cdot X$, $x \in \Gamma(t)$, so R is a measure of the deviation in the perturbed free boundary from the unperturbed position.

To (2.2)-(2.5) we add the initial conditions

$$\mathbf{X}|_{t=0} = 0, \quad \mathbf{X}_t|_{t=0} = \mathbf{U}_0, \quad \mathrm{div}\ \mathbf{U}_0 = 0. \tag{2.6}$$

The (2.2)-(2.6) treatment describes the evolution of small perturbations in an inhomogeneous ideal incompressible liquid having a free boundary in the linear approximation.

3. Planar-Layer Motional Stability. We consider a basic two-dimensional flow defined by

$$x = \xi, \quad y = \eta - s(t), \quad s(0) = 0, \quad p(\eta, t) = (s'' - g) \int_0^{\eta} \rho_0(z)\, dz + p_0(t),$$
$$\xi \in (-\infty, +\infty), \quad \eta \in [0, \eta_0], \quad g = (0, -g).$$

This solution corresponds to the motion of an infinite planar layer bounded by horizontal free boundaries. At the upper boundary, we are given the pressure $p_1(t)$, and at the lower one, $p_0(t)$. The function s(t) should satisfy

$$s'' - g = \frac{p_1(t) - p_0(t)}{\int_0^{\eta_0} \rho_0\, dz}.$$

Then the layer moves as a result of the difference in pressures above and below, and also under gravity. The final formula for $p(\eta, t)$ for a layer having free boundaries is

$$p(\eta, t) = \frac{(p_1(t) - p_0(t)) \int_0^{\eta} \rho_0\, dz}{\int_0^{\eta_0} \rho_0\, dz + p_0(t)}.$$

We use (2.2)-(2.6) to examine the stability. We render the variables dimensionless by taking the scale factors for the spatial variables, time, velocity, density, and pressure respectively as η_0, $\sqrt{\eta_0/g}$, $\sqrt{g\eta_0}$, $\rho_0(\eta_0)$, $g\eta_0\rho_0(\eta_0)$.

Let $\Psi(\xi, \eta, t)$ be a sufficiently smooth function such that $\Psi_\xi = -Y$, $\Psi_\eta = X$. Then we get the following treatment:

$$\Psi_{\xi\xi tt} + \Psi_{\eta\eta tt} + \rho_{0\eta}\frac{\Psi_{\eta tt}}{\rho_0} + Q_{0\xi}\frac{a}{\rho_0} + \rho_{0\eta} a \frac{\Psi_{\xi\xi}}{\rho_0} = 0, \quad 0 < \eta < 1,$$
$$\rho_0(\Psi_{\eta tt} + a\Psi_{\xi\xi}) = -\mathrm{Bo}\Psi_{\xi\xi\xi\xi}, \quad \eta = 1,$$
$$\rho_0(\Psi_{\eta tt} + a\Psi_{\xi\xi}) = \mathrm{Bo}\Psi_{\xi\xi\xi\xi}, \quad \eta = 0,$$

$$\Psi|_{t=0} = 0, \quad \Psi_t|_{t=0} = \Psi_0(\xi, \eta)$$

in which ($\Psi_0(\xi, \eta)$ is a given function, $\mathrm{Bo} = \sigma/(g\eta_0^2\rho_0(\eta_0))$ is the Bond number, and $a = s'' - g$). If g = 0, the equations remain as before, and only $a = s''/g_0$, where g_0 is the characteristic acceleration. Coefficient a is expressed in terms of the dimensionless quantities as

$$a = \frac{p_1(t) - p_0(t)}{\int_0^{1} \rho_0\, dz}.$$

We assume that the density of the liquid particles is not perturbed at the initial instant and that the layer moves with equal acceleration and that the liquid stratification is exponential, i.e., $Q_0 = 0$, $a = \text{const}$, $\rho_{0\eta}/\rho_0 = r = \text{const}$.

The sign of a characterizes the pressure difference between the upper and lower boundaries, while the sign of r indicates the direction of the density gradient.

We seek a solution as the series

$$\sum_{n,k} \Psi_{nk}(\xi, \eta, t) = \sum_{n,k} \Phi_{nk}(\eta) \exp(i(n\xi + \alpha_{nk} t)).$$

The characteristic instability parameters (complex quantities) α_{nk} are determined from the corresponding spectral treatment. If the imaginary part of at least one of the α_{nk} is negative, the solution will contain an exponentially increasing mode, and this is a sign of instability. We subsequently put $\alpha = \alpha_{nk}$, omitting the subscripts.

We write the spectral problem as

$$\begin{aligned} &\alpha^2\Phi_{\eta\eta} + r\alpha^2\Phi_\eta + (arn^2 - n^2\alpha^2)\Phi = 0, \quad 0 < \eta < 1, \\ &\alpha^2\Phi_\eta = n^4\mathrm{Bo}\ \Phi - n^2 a\Phi, \quad \eta = 1, \\ &\alpha^2\Phi_\eta = -n^4\mathrm{Bo}\ \Phi \exp(r) - n^2 a\Phi, \quad \eta = 0. \end{aligned}$$

Let $a = 0$. We substitute the solution to the first equation here in the boundary conditions to get an algebraic system for C_1 and C_2:

$$C_1(\alpha^2\gamma_1 + n^4\mathrm{Bo}\exp(r)) + C_2(\alpha^2\gamma_2 + n^4\mathrm{Bo}\exp(r)) = 0,$$

$$C_1(\alpha^2\gamma_1 - n^4\mathrm{Bo})\exp(\gamma_1) + C_2(\alpha^2\gamma_2 - n^4\mathrm{Bo})\exp(\gamma_2) = 0.$$

We equate the determinant of the resulting system to zero and get an equation for α, where calculations show that $A\alpha^4 + B\alpha^2 + C = 0$, where calculations show that $A < 0$, $B > 0$, $C < 0$, Vietta's theorem gives that all the roots α are real. Then for $a = 0$, the motion is neutrally stable, when the layer falls only under the action of gravity ($s'' = g$).

Now let $a \neq 0$. We make the substitution $\Phi = y \exp(-r\eta/2)$, to get

$$y'' + Iy = 0, \quad 0 < \eta < 1; \tag{3.1}$$

$$\begin{aligned} &y' + I_1 y = 0, \quad \eta = 1; \\ &y' + I_2 y = 0, \quad \eta = 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} I = -r^2/4 + ar(n/\alpha)^2 - n^2, \quad I_1 = -r/2 + (n/\alpha)^2 a - n^4\mathrm{Bo}/\alpha^2, \\ I_2 = -r/2 + (n/\alpha)^2 a + n^4\mathrm{Bo}\exp(r)/\alpha^2. \end{aligned} \tag{3.3}$$

Let $I = 0$, i.e., $\alpha^2 = n^2ar/(r^2/4 + n^2)$. Then the general solution to (3.1) is $y = C_1\eta + C_2$. Substitution into the boundary conditions gives a system for C_1 and C_2, which has nontrivial solutions if $I_1 - I_2 - I_1 I_2 = 0$. We substitute detailed expressions for I_1 and I_2 to get a biquadratic equation for α. The requirement that the roots of this equation should satisfy $\alpha_{1,2}^2 = n^2ar/(r^2/4 + n^2)$, is a constraint on the parameters.

Let $I \neq 0$. We substitute the general solution to (3.1) into the boundary conditions to get an equation for α:

$$(I + I_1 I_2)\sin\sqrt{I} = \sqrt{I}(I_1 - I_2)\cos\sqrt{I}. \tag{3.4}$$

If we neglect surface tension (Bo = 0), we have

$$\alpha = \sqrt{|an|}\exp(i\pi m/2), \qquad m = 0, 1, 2, 3; \tag{3.5}$$

$$\alpha = \pm n\sqrt{ar/(r^2/4 + n^2 + (\pi k)^2)}, \quad \text{if} \quad ar > 0; \tag{3.6}$$

$$\alpha = \pm in\sqrt{|ar|/(r^2/4 + n^2 + (\pi k)^2)}, \quad \text{if} \quad ar < 0, \tag{3.7}$$

$$k = 0, \pm 1, \pm 2, \ldots$$

The first group of values for α in (3.5) does not intersect with the second group (3.6) and (3.7). With any relation between the parameters, the motion is unstable on account of the first group.

This conclusion may be compared with the [2] results, which dealt with the stability of motion for a planar layer of homogeneous liquid. Although the perturbations were taken as different, the linearity of the treatments allows one to compare them.

When Bo = 0, the first group of α corresponds to the values obtained in [2] for a homogeneous liquid. These α reflect the growth of surface waves, which are not related to inhomogeneity in the liquid and are due to interphase boundaries. The stratification led to the occurrence of a further set: a denumerable series of values for α that expresses the rise rates for the so-called internal waves. There are two justifications for the name. In that case, not only do internal factors participate in generating the waves, but also the waves perturb only the interior part of the layer and do not affect the boundaries and thus the external medium.

In [2] it was also shown that when there are capillary forces, the surface tension stabilizes the instability in modes with sufficiently short wavelengths. However, in our case, (3.4) with $n \to \infty$ gives

$$\alpha \approx \pm\sqrt{ar}, \quad ar > 0, \qquad \alpha \approx \pm i\sqrt{|ar|}, \quad ar < 0.$$

In that first approximation, the surface tension does not stabilize sufficiently short-wave internal waves.

If we supply solid walls at the lower and upper boundaries, we get the equation $\sin\sqrt{I} = 0$ for α.

The values of α are defined by (3.6) and (3.7), in which a is expressed as $a = s''/g - 1$, so in this case the motion is neutrally stable for $ar > 0$ and unstable for $ar < 0$. As the surface waves are suppressed here by the solid walls, it is now more evident that the (3.6) and (3.7) series correspond to internal waves. They evidently in this approximation are not related to surface waves, since the absence of surface waves does not alter (3.6) and (3.7), while the absence of internal waves in such a case for a homogeneous liquid [2] does not alter the character of the formulas corresponding in [2] to (3.5). We note that the internal waves do not perturb the free boundaries in the linear theory.

We now compare these results with conclusions from papers dealing with the stability at rest for an inhomogeneous layer. In [3], stability was examined for a thin layer of liquid between two semiinfinite media having constant density. The wavelength of the perturbations was assumed large by comparison with the layer thickness. It was found that if the acceleration direction for the external mass forces is opposite to that of the density gradient, there exists a denumerable infinite number of modes as internal waves for each wave number. The formulas for α correspond to those obtained in our case with g replaced by a.

This correspondence and the physical reasons for the instability indicate that the characteristic internal-wave instability parameters are dependent on the boundary conditions only via the pressure distribution in the basic unperturbed motion. With the same pressure distribution, the layer may adjoin a gas (basic example here) or solid walls, or else liquid ones [3] (at least in the thin-layer approximation), and the results will be identical. From that viewpoint, linear internal waves are similar to modes in nonlinear systems on account of a certain independence from the external factors.

In [4, 5], instability was considered in a two-layer or continuously stratified liquid by means of the analogy with stability theory for hamiltonian finite-dimensional systems. Let Φ be the potential for the external mass forces, with the basic state hydrostatic equilibrium in a certain region having a fixed boundary, and ρ_0 the equilibrium density function. We restrict the class of permissible motions by means of the condition $Q_0(\xi) = 0$ and an additional requirement (a generalized treatment of the potentiality of the motion) to show that if $d\Phi(\rho_0)/d\rho_0 < 0$, everywhere in the region occupied by the liquid, then the equilibrium is stable in the mean-squares. If everywhere $d\Phi(\rho_0)/d\rho_0 > 0$, the basic state is unstable. This corresponds to the [3] results and ours.

In [6], a similar treatment was considered for internal waves in an unbounded planar channel at rest bounded by solid walls. The stratification of the liquid was exponential and $r < 0$, while the initial data were finite and certain additional conditions were introduced. In our case, this corresponds to neutral stability ($a = -1$, $ar > 0$). An interesting point is that the solution for $t \to +\infty$ stabilizes to steady-state plane–parallel flow if the accelerations of the particles differ from zero at the initial instant. Evidently, an analogous effect arises for a moving layer with solid walls.

To sum up, we can say that internal waves do not increase exponentially if the pressure is higher in the denser part. This corresponds to Taylor's classical result on the instability of the interface between liquids differing in density [7]: the interface boundary is unstable when the acceleration of the light liquid is directed into the heavier one (the gravitational force is replaced by the accelerated motion of the coordinate system) and is stable if the converse applies.

I am indebted to V. K. Andreev for assistance.

REFERENCES

1. V. K. Andreev, Stability in the Transient-State Motion of a Liquid Having a Free Boundary [in Russian], Nauka, Novosibirsk (1992).
2. V. M. Kuznetsov and E. N. Sher, "Stability in the flow of an ideal incompressible liquid in a band and in a ring," Prikl. Mekh. Tekh. Fiz., No. 2, 66-73 (1964).
3. J. N. Hunt, "Taylor instability in a thin fluid layer," Appl. Sci. Res., **10**, 45-58 (1961).
4. V. A. Vladimirov, "Instability of the equilibrium of an inhomogeneous liquid in cases where the potential energy is not minimal," Prikl. Mat. Mekh., **52**, No. 3, 415-422 (1988).
5. S. Ya. Belov and V. A. Vladimirov, "An example of inverting Lagrange's theorem in the hydrodynamics of a two-layer liquid," in: Dynamics of Continuous Media, Coll. [in Russian], Inst. Gidrodinamiki, Sib. Otd. AN SSSR, Issue 84 (1988), pp. 21-27.
6. S. A. Gabov and A. G. Sveshnikov, Linear Treatments of Nonstationary Internal Waves [in Russian], Nauka, Moscow (1990).
7. G. Birkhoff, Helmholz and Taylor Instabilities: Hydrodynamic Instability [Russian translation], Mir, Moscow (1964), pp. 68-94.